

On the simplicity of Kac modules for the restricted Lie superalgebra $gl(m, n)$

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1 Introduction

Let $\mathfrak{g} = gl(m, n)$ be the general linear Lie superalgebra over a field \mathbb{F} with $\text{char.}\mathbb{F} = p > 2$. Then \mathfrak{g} is a restricted Lie superalgebra. The Kac module of \mathfrak{g} is a finite dimensional module (see Sec.2) induced from a simple \mathfrak{g}_0 -module. In the characteristic zero case, the simplicity of Kac modules is determined by typical weights (see [5]) which were defined using a symmetric bilinear form on H^* , where H is the maximal torus of \mathfrak{g} consisting of diagonal matrices. In the case that $\text{char.}\mathbb{F} = p > 2$, the simplicity of the Kac modules was studied in [9, 11]. In the restricted case, it was shown in [11, Th.2.2] that the simplicity of the Kac modules is determined by a polynomial. But the conclusion that this polynomial coincides with the polynomial $P(\lambda)$ defined in complex number case ([5]) modulo p is stated without proof([11, Prop. 2.1]). In more general cases, the simplicity of the Kac modules was studied in [9], in which one has to assume $p > nm$ to determine an analogous polynomial.

The main goal of the present paper is to present a characteristic free approach to determine the above-mentioned polynomial. Then we study its application to the nonrestricted simple modules for \mathfrak{g} . The paper is arranged as follows. Sec. 2 is the preliminaries. In Sec. 3 we discuss the simplicity of the Kac modules. In Sec. 4 we prove the main theorem. By applying the main theorem we show in Sec. 5 that, under certain conditions, the χ -reduced enveloping superalgebras $u_\chi(\mathfrak{g})$ and $u_\chi(\mathfrak{g}_0)$ are Morita equivalent.

2 Preliminaries

Let $\mathfrak{g} = gl(m, n)$ be the general linear Lie superalgebra (See [5]). Then \mathfrak{g} has a standard basis consisting of matrices $\{e_{ij} | 1 \leq i, j \leq m+n\}$. Set $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$, where

$$\begin{aligned}\mathcal{I}_0 &= \{(i, j) | 1 \leq i < j \leq m \text{ or } m+1 \leq i < j \leq m+n\}, \\ \mathcal{I}_1 &= \{(i, j) | 1 \leq i \leq m < j \leq m+n\}.\end{aligned}$$

We denote e_{ji} with $j > i$ by f_{ij} . Then we get $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, where

$$\mathfrak{g}_1 = \langle e_{ij} | (i, j) \in \mathcal{I}_1 \rangle \quad \mathfrak{g}_{-1} = \langle f_{ij} | (i, j) \in \mathcal{I}_1 \rangle.$$

Let \mathfrak{g}^+ (resp. \mathfrak{g}^-) be the subalgebra $\mathfrak{g}_{\bar{0}} + \mathfrak{g}_1$ (resp. $\mathfrak{g}_{\bar{0}} + \mathfrak{g}_{-1}$) of \mathfrak{g} . The parity of the basis elements is given by

$$\bar{e}_{ij} = \bar{f}_{ij} = \begin{cases} \bar{0}, & \text{if } (i, j) \in \mathcal{I}_0 \text{ or } i = j \\ \bar{1}, & \text{if } (i, j) \in \mathcal{I}_1. \end{cases}$$

Let $H = \langle e_{ii} | 1 \leq i \leq m+n \rangle$, and let T be the linear algebraic group consisting of $(m+n) \times (m+n)$ invertible diagonal matrices. Then we have $\text{Lie}(T) = H$. Let $\Lambda =: X(T) = \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 + \cdots + \mathbb{Z}\epsilon_{m+n}$. The set of positive roots of \mathfrak{g} relative to T is $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$, where

$$\Phi_0^+ = \{\epsilon_i - \epsilon_j | (i, j) \in \mathcal{I}_0\}, \quad \Phi_1^+ = \{\epsilon_i - \epsilon_j | (i, j) \in \mathcal{I}_1\}.$$

Let

$$\rho_0(m, n) = 1/2 \sum_{\alpha \in \Phi_0^+} \alpha, \quad \rho_1(m, n) = 1/2 \sum_{\alpha \in \Phi_1^+} \alpha,$$

and set $\rho(m, n) =: \rho_0(m, n) - \rho_1(m, n) \in \Lambda$.

Denote by N^+ (resp. N^-) the Lie sub-superalgebra of \mathfrak{g} spanned by the elements $e_{ij}, (i, j) \in \mathcal{I}$ (resp. $f_{ij}, (i, j) \in \mathcal{I}$). By the PBW theorem ([1]) we have the triangular decomposition of $U(\mathfrak{g})$:

$$U(\mathfrak{g}) \cong U(N^-) \otimes U(H) \otimes U(N^+).$$

Let $h(U(\mathfrak{g}))$ be the set of all homogeneous elements in $U(\mathfrak{g})$. For each $x \in h(U(\mathfrak{g}))$, we have a derivation $[x, -]$ on $U(\mathfrak{g})$ defined by

$$[x, y] = xy - (-1)^{\bar{x}\bar{y}}yx, \quad y \in h(U(\mathfrak{g})).$$

It is easy to see that

$$[x, y_1 \cdots y_t] = \sum_{i=1}^t (-1)^{\bar{x} \sum_{k=1}^{i-1} \bar{y}_k} y_1 \cdots [x, y_i] \cdots y_t,$$

for $y_1, \dots, y_t \in h(U(\mathfrak{g}))$.

Let $\check{\epsilon}_i$ be the 1-psg: $G_m \longrightarrow T$ such that each $t \in G_m$ is mapped into a diagonal matrix with all entries equal to 1 but the i th equal to t if $i \leq m$, and t^{-1} if $i > m$. Then the 1-psg's $\check{\epsilon}_i$ form a \mathbb{Z} -basis of $Y(T)$. The nondegenerate paring([3]):

$$X(T) \times Y(T) \longrightarrow \mathbb{Z} : (\lambda, \mu) \mapsto \langle \lambda, \mu \rangle$$

induces a symmetric bilinear form on Λ defined by

$$(\epsilon_i, \epsilon_j) = \langle \epsilon_i, \check{\epsilon}_j \rangle = \begin{cases} \delta_{ij}, & i \leq m \\ -\delta_{ij}, & i > m. \end{cases}$$

Assume the Lie superalgebra \mathfrak{g} is defined over a field \mathbb{F} . We identify H^* with $\Lambda \otimes_{\mathbb{Z}} \mathbb{F}$. Then the bilinear form above is extended naturally to H^* . In case $\text{char.}\mathbb{F} = 0$, this is exactly the one given in [5]. Suppose $\text{char.}\mathbb{F} = p > 0$. For each $\lambda \in \Lambda = X(T)$, the tangent map $d\lambda: H \longrightarrow \mathbb{F}$, by [4, 1.2], is a linear map satisfying $d\lambda(h^{[p]}) = (d\lambda(h))^p$ for all $h \in H$. By identifying $d\check{\epsilon}_i(1)$ with e_{ii} if $i \leq m$ and $-e_{ii}$ if $i > m$, we see that $d\lambda$ is exactly $\lambda \otimes 1 \in \Lambda \otimes_{\mathbb{Z}} \mathbb{F} = H^*$. For each $\lambda \in \Lambda$, we write $\lambda \otimes 1 \in H^*$ also as λ .

Define the polynomial $f_{m,n}(\lambda)$ on H^* by

$$f_{m,n}(\lambda) = \Pi_{\alpha \in \Phi_1^+}(\lambda + \rho(m, n), \alpha), \lambda \in H^*.$$

$\lambda \in H^*$ is referred to as *typical* if $f_{m,n}(\lambda) \neq 0$

3 The simplicity of Kac modules

Let \mathfrak{g} be the Lie superalgebra $gl(m, n)$ over a field \mathbb{F} . Let $U(\mathfrak{g})$ (resp. $U(\mathfrak{g}^+)$; $U(\mathfrak{g}_0)$; $U(\mathfrak{g}_{-1})$) be the universal enveloping superalgebra of \mathfrak{g} (resp. \mathfrak{g}^+ ; \mathfrak{g}_0 ; \mathfrak{g}_{-1})(see [1]).

Let M be a $U(\mathfrak{g}_0)$ -module. For $\mu \in H^*$, define the μ -weight space of M by

$$M_\mu = \{x \in M | hx = \mu(h)x \text{ for all } h \in H\}.$$

For any positive odd root $\alpha = \epsilon_i - \epsilon_j \in \Phi_1^+$, $(i, j) \in \mathcal{I}_1$, let $h_\alpha = [e_{ij}, f_{ij}] = e_{ii} + e_{jj}$. It is easy to check that $\mu(h_\alpha) = (\mu, h_\alpha)$, for any $\mu \in \Lambda, \alpha \in \Phi_1^+$. Therefore we get $h_\alpha x = (\alpha, \mu)x$ for any $x \in M_\mu$. If $x \in M_\mu$, then

$$f_{ij}x \in M_{\mu - (\epsilon_i - \epsilon_j)} \quad \text{and} \quad e_{ij}x \in M_{\mu + \epsilon_i - \epsilon_j}$$

for any $(i, j) \in \mathcal{I}_0$. A nonzero vector $v^+ \in M_\mu$ is said to be maximal if $e_{ij}v^+ = 0$ for all $(i, j) \in \mathcal{I}_0$.

Let $M_0(\lambda)$ be a simple $U(\mathfrak{g}_0)$ -module generated by a maximal vector of weight $\lambda \in H^*$. We can view $M_0(\lambda)$ as a $U(\mathfrak{g}^+)$ -module by letting \mathfrak{g}_1 act trivially on it. Then the induced $U(\mathfrak{g})$ -module

$$K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^+)} M_0(\lambda)$$

is called a Kac module. In case $\mathbb{F} = \mathbb{C}$, [5, Prop. 2.9] says that $K(\lambda)$ is simple if and only if λ is typical.

Note: In the case that $\text{char.}\mathbb{F} > 0$, the maximal vector in the $U(\mathfrak{g}_0)$ -module $M_0(\lambda)$ and hence its weight $\lambda \in H^*$ may not be unique.

Let $M_0(\lambda)$ be a simple $U(\mathfrak{g}_0)$ -module generated by a maximal vector of weight $\lambda \in H^*$. If $\text{char.}\mathbb{F} = p > 0$, then $M_0(\lambda)$ is finite dimensional by the Jacobson's theorem(see [7, Th. 2.4]), so is the Kac module $K(\lambda)$.

By definition, we have

$$K(\lambda) \cong U(\mathfrak{g}_{-1}) \otimes_{\mathbb{F}} M_0(\lambda)$$

as $U(\mathfrak{g}_{-1})$ -modules.

Let us define an order on the set \mathcal{I}_1 as follows:

$$(i, j) \prec (s, t) \quad \text{if and only if} \quad j > t \quad \text{or} \quad j = t \quad \text{but} \quad i < s.$$

We write $(i, j) \preceq (s, t)$ if $(i, j) \prec (s, t)$ or $(i, j) = (s, t)$. Define $f_{ij} \prec f_{st}$ if $(i, j) \prec (s, t)$. For each subset $I \subseteq \mathcal{I}_1$, let f_I denote the product $\prod_{(i,j) \in I} f_{ij}$ in this order. In particular, $f_\emptyset = 1$. Then it is clear that $K(\lambda)$ has a basis

$$f_I \otimes v_i, I \subseteq \mathcal{I}_1, i = 1, \dots, s,$$

where v_1, \dots, v_s is a basis of $M_0(\lambda)$. Note that $f_{ij}f_{st} = -f_{st}f_{ij} \in U(\mathfrak{g}_{-1}) \subseteq U(\mathfrak{g})$, for any $(i, j), (s, t) \in \mathcal{I}_1$. Then it is easily seen that, by multiplying appropriate f'_{ij} 's ($(i, j) \in \mathcal{I}_1$) to any nonzero element $x \in K(\lambda)$, one obtains $f_{\mathcal{I}_1} \otimes v$ with $0 \neq v \in M_0(\lambda)$.

It is easy to check that, for any $(i, j) \in \mathcal{I}_0$,

$$(*) \quad f_{ij}f_{\mathcal{I}_1} = f_{\mathcal{I}_1}f_{ij}, \quad e_{ij}f_{\mathcal{I}_1} = f_{\mathcal{I}_1}e_{ij}.$$

For each subset $I \subseteq \mathcal{I}_1$, we denote by e_I the product $\prod_{(i,j) \in I} e_{ij}$ in the reversed order of \mathcal{I}_1 . For each $(i, j) \in \mathcal{I}_1$, let

$$> (i, j) (\text{resp. } \geq (i, j); < (i, j); \leq (i, j);)$$

denote the subset

$$\begin{aligned} & \{(s, t) \in \mathcal{I}_1 | (s, t) \succ (i, j)\} \\ & (\text{resp. } \{(s, t) \in \mathcal{I}_1 | (s, t) \succeq (i, j)\}; \\ & \{(s, t) \in \mathcal{I}_1 | (s, t) \prec (i, j)\}; \\ & \{(s, t) \in \mathcal{I}_1 | (s, t) \preceq (i, j)\}). \end{aligned}$$

For $(i, j), (s, t) \in \mathcal{I}_1$ with $(i, j) \prec (s, t)$, we denote by $((i, j), (s, t))$ the subset $\{(i', j') \in \mathcal{I}_1 | (i, j) \prec (i', j') \prec (s, t)\}$.

Let us write $e_{\mathcal{I}_1}f_{\mathcal{I}_1} \in U(\mathfrak{g})$ in terms of the triangular decomposition of $U(\mathfrak{g})$ (see Sec.2):

$$e_{\mathcal{I}_1}f_{\mathcal{I}_1} = f(h) + \sum u_i^- u_i^0 u_i^+, u_i^\pm \in U(N^\pm), f(h), u_i^0 \in U(H).$$

Note that $U(\mathfrak{g})$ is a T -module under the adjoint action. Denote by $\text{wt}(u)$ the weight of a weight vector $u \in U(\mathfrak{g})$. Then since $\text{wt}(e_{\mathcal{I}_1} f_{\mathcal{I}_1}) = 0$, so that $\text{wt}(u_i^+) = -\text{wt}(u_i^-)$ we get $u_i^+ \in \mathbb{F}$ if and only if $u_i^- \in \mathbb{F}$.

Let v_λ be a maximal vector in $M_0(\lambda) \subseteq K(\lambda)$. Then we get

$$e_{\mathcal{I}_1} f_{\mathcal{I}_1} v_\lambda = f(h) v_\lambda = f(h)(\lambda) v_\lambda.$$

The following proposition was proved in [6] in the case $\mathbb{F} = \mathbb{C}$, and proved in [11] in the restricted case.

Proposition 3.1. *Let $\lambda \in H^*$. Then $K(\lambda)$ is simple if and only if $f(h)(\lambda) \neq 0$.*

Proof. By the formula (*) above, the subspace $f_{\mathcal{I}_1} \otimes M_0(\lambda) \subseteq K(\lambda)$ is also a simple $U(\mathfrak{g}_0)$ -module, and which is clearly annihilated by \mathfrak{g}_{-1} . It follows that

$$U(\mathfrak{g}_1) f_{\mathcal{I}_1} \otimes M_0(\lambda) = U(\mathfrak{g}) f_{\mathcal{I}_1} \otimes M_0(\lambda),$$

is a $U(\mathfrak{g})$ -submodule of $K(\lambda)$.

Suppose $K(\lambda)$ is simple. Then $U(\mathfrak{g}_1) f_{\mathcal{I}_1} \otimes M_0(\lambda) = K(\lambda)$. Since $\dim U(\mathfrak{g}_1) = \dim U(\mathfrak{g}_{-1})$, $K(\lambda)$ has a basis consisting of elements

$$e_I m_s, \quad I \subseteq \mathcal{I}_1, s = 1, \dots, r,$$

where m_1, \dots, m_r is a basis of $f_{\mathcal{I}_1} \otimes M_0(\lambda)$. Choose $m_1 = f_{\mathcal{I}_1} \otimes v_\lambda$, where $v_\lambda \in M_0(\lambda)$ is a maximal vector of weight λ . Then we have

$$0 \neq e_{\mathcal{I}_1} f_{\mathcal{I}_1} \otimes v_\lambda = 1 \otimes f(h)(\lambda) v_\lambda,$$

and hence $f(h)(\lambda) \neq 0$.

Suppose $f(h)(\lambda) \neq 0$. Assume $K = K_{\bar{0}} \oplus K_{\bar{1}}$ is a nonzero submodule of $K(\lambda)$. Let $x \in h(K)$ be a nonzero vector. Apply appropriate f_{ij} 's to x to obtain $f_{\mathcal{I}_1} \otimes m \in K$ with $0 \neq m \in M_0(\lambda)$. From the formula (*) above it follows that $f_{\mathcal{I}_1} \otimes v_\lambda \in K$, since $M_0(\lambda)$ is a simple $U(\mathfrak{g}_0)$ -module. Then we get

$$e_{\mathcal{I}_1} f_{\mathcal{I}_1} \otimes v_\lambda = 1 \otimes f(h)(\lambda) v_\lambda \in K,$$

so that $v_\lambda \in K$. This gives $K = K(\lambda)$, and hence $K(\lambda)$ is simple. \square

4 The main theorem

In this section we give a characteristic free approach to determine the polynomial $f(h)(\lambda)$.

Lemma 4.1. *Let $1 \leq i \leq m$. Then $e_{i,m+n} f_{>(i,m+n)} v_\lambda = 0$.*

Proof. For each $(s, t) \succ (i, m+n)$, we have

$$[e_{i,m+n}, f_{st}] = \begin{cases} e_{t,m+n}, & \text{if } i = s, t < m+n \\ e_{is}, & \text{if } i < s, t = m+n \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} e_{i,m+n} f_{>(i,m+n)} v_\lambda &= [e_{(i,m+n)}, f_{>(i,m+n)}] v_\lambda \\ &= \sum_{f_{st} \succ f_{i,m+n}} (-1)^{\alpha_{st}} f_{((i,m+n),(s,t))} [e_{i,m+n}, f_{st}] f_{>(s,t)} v_\lambda \\ &= \sum_{s>i, t=m+n} (-1)^{\alpha_{st}} f_{((i,m+n),(s,t))} e_{is} f_{>(s,m+n)} v_\lambda \\ &\quad + \sum_{s=i, t<m+n} (-1)^{\alpha_{st}} f_{((i,m+n),(s,t))} e_{t,m+n} f_{>(s,t)} v_\lambda, \end{aligned}$$

where $\alpha_{st} \in \mathbb{Z}_2$. Note that the second summation is equal to zero, since $e_{t,m+n}$ commutes with all $f_{ij} ((i, j) \in \mathcal{I}_1)$ with $f_{ij} \succ f_{st}$.

We claim that the first summation is also equal to zero. In fact, we have, in the case where $s > i, t = m+n$,

$$\begin{aligned} e_{is} f_{>(s,m+n)} v_\lambda &= [e_{is}, f_{>(s,m+n)}] v_\lambda \\ &= \sum_{j=m+n-1}^{m+1} f_{((s,m+n),(i,j))} [e_{is}, f_{ij}] f_{>(i,j)} v_\lambda \\ &= \sum_{j=m+n-1}^{m+1} f_{((s,m+n),(i,j))} f_{sj} f_{>(i,j)} v_\lambda = 0, \end{aligned}$$

where the last equality is given by the fact that $f_{sj} \succ f_{ij}$. □

Theorem 4.2. For each $\lambda \in H^*$, we have $f(h)(\lambda) = f_{m,n}(\lambda)$.

Proof. We have

$$\begin{aligned} e_{\mathcal{I}_1} f_{\mathcal{I}_1} v_\lambda &= e_{>(1,m+n)} (e_{1,m+n} f_{1,m+n}) f_{>(1,m+n)} v_\lambda \\ &= e_{>(1,m+n)} (e_{11} + e_{m+n,m+n}) f_{>(1,m+n)} v_\lambda \\ &\quad - e_{>(1,m+n)} f_{1,m+n} e_{1,m+n} f_{>(1,m+n)} v_\lambda \\ (\text{Using Lemma 4.1}) &= e_{>(1,m+n)} (e_{11} + e_{m+n,m+n}) f_{>(1,m+n)} v_\lambda \\ &= (\lambda + \alpha_1) (e_{11} + e_{m+n,m+n}) e_{>(1,m+n)} f_{>(1,m+n)} v_\lambda \\ &= (\lambda + \alpha_1, \epsilon_1 - \epsilon_{m+n}) e_{>(1,m+n)} f_{>(1,m+n)} v_\lambda, \end{aligned}$$

where $\lambda + \alpha_1$ is the weight of $f_{>(1,m+n)} v_\lambda$.

Using Lemma 4.1, we compute $e_{>(1,m+n)} f_{>(1,m+n)} v_\lambda$ in a similar way. Continue the process, we get

$$e_{\mathcal{I}_1} f_{\mathcal{I}_1} v_\lambda = \prod_{i=1}^k (\lambda + \alpha_i) (e_{ii} + e_{m+n,m+n}) e_{>(k,m+n)} f_{>(k,m+n)} v_\lambda$$

$$\begin{aligned}
&= \dots \\
&= \Pi_{i=1}^m (\lambda + \alpha_i) (e_{ii} + e_{m+n, m+n}) e_{\geq(1, m+n-1)} f_{\geq(1, m+n-1)} v_\lambda \\
&= \Pi_{i=1}^m (\lambda + \alpha_i, \epsilon_i - \epsilon_{m+n}) e_{\geq(1, m+n-1)} f_{\geq(1, m+n-1)} v_\lambda.
\end{aligned}$$

For each $1 \leq i \leq m$, it is easily seen that the weight of $f_{>(i, m+n)} v_\lambda$ is

$$\lambda + \alpha_i = \lambda - 2\rho_1(m, n) + \sum_{k=1}^i (\epsilon_k - \epsilon_{m+n}).$$

By a straightforward computation we have, for each $1 \leq i \leq m$,

$$(1) \quad (-\rho_0(m, n) - \rho_1(m, n) + \sum_{k=1}^i (\epsilon_k - \epsilon_{m+n}), \epsilon_i - \epsilon_{m+n}) = 0.$$

Applying the formula (1), we have

$$(\lambda + \alpha_i, \epsilon_i - \epsilon_{m+n}) = (\lambda + \rho(m, n), \epsilon_i - \epsilon_{m+n})$$

for any $1 \leq i \leq m$, which gives

$$f(h)(\lambda) = \Pi_{k=1}^m (\lambda + \rho(m, n), \epsilon_k - \epsilon_{m+n}) e_{\geq(1, m+n-1)} f_{\geq(1, m+n-1)} v_\lambda.$$

We now prove the theorem by induction on n . The case $n = 1$ follows immediately from the above equation. Assume the case $n - 1$ and consider the case n . Note that

$$\rho(m, n) = \rho(m, n - 1) + \frac{1}{2} \left[\sum_{k>m} (\epsilon_k - \epsilon_{m+n}) - \sum_{k \leq m} (\epsilon_k - \epsilon_{m+n}) \right].$$

By a short computation we have

$$\left(\sum_{k>m} (\epsilon_k - \epsilon_{m+n}) - \sum_{k \leq m} (\epsilon_k - \epsilon_{m+n}), \epsilon_i - \epsilon_j \right) = 0$$

for all $i \leq m < j < m + n$, so that

$$(\lambda + \rho(m, n - 1), \epsilon_i - \epsilon_j) = (\lambda + \rho(m, n), \epsilon_i - \epsilon_j).$$

Then we have by the induction hypothesis that

$$\begin{aligned}
f(h)(\lambda) &= \Pi_{k=1}^m (\lambda + \rho(m, n), \epsilon_k - \epsilon_{m+n}) \Pi_{i \leq m < j < m+n} (\lambda + \rho(m, n - 1), \epsilon_i - \epsilon_j) \\
&= \Pi_{(i, j) \in \mathcal{I}_1} (\lambda + \rho(m, n), \epsilon_i - \epsilon_j) = f_{m, n}(\lambda).
\end{aligned}$$

□

Corollary 4.3. *Let $\mathfrak{g} = gl(m, n)$ be defined over \mathbb{F} with $\text{char.}\mathbb{F} = p > 2$. Then $K(\lambda)$ is simple if and only if $\Pi_{(i, j) \in \mathcal{I}_1} (\lambda + \rho(m, n), \epsilon_i - \epsilon_j) \neq 0$.*

5 Applications of the main theorem

In this section, we assume $\mathfrak{g} = gl(m, n)$ is defined over a field \mathbb{F} of characteristic $p > 0$. We abbreviate $\rho(m, n)$ to ρ . Then \mathfrak{g} is a restricted Lie superalgebra (see[1]) with the p -map $[p]$ the p th power map.

By [8, 10], each simple \mathfrak{g} -module $M = M_{\bar{0}} \oplus M_{\bar{1}}$ affords a p -character $\chi \in \mathfrak{g}_{\bar{0}}^*$ such that

$$(x^p - x^{[p]} - \chi(x)^p)v = 0$$

for all $x \in \mathfrak{g}_{\bar{0}}, v \in M$. Let $u_{\chi}(\mathfrak{g})$ (resp. $u_{\chi}(\mathfrak{g}_{\bar{0}})$; $u_{\chi}(\mathfrak{g}^+)$; $u_{\chi}(\mathfrak{g}^-)$) be the quotient superalgebra of $U(\mathfrak{g})$ (resp. $U(\mathfrak{g}_{\bar{0}})$; $U(\mathfrak{g}^+)$; $U(\mathfrak{g}^-)$) by its \mathbb{Z}_2 -graded two-sided ideal generated by the central elements

$$x^p - x^{[p]} - \chi(x)^p, x \in \mathfrak{g}_{\bar{0}}.$$

Then M is a $u_{\chi}(\mathfrak{g})$ -module. The superalgebras $u_{\chi}(\mathfrak{g}_{\bar{0}})$, $u_{\chi}(\mathfrak{g}^+)$ and $u_{\chi}(\mathfrak{g}^-)$ are all viewed canonically as subalgebras of $u_{\chi}(\mathfrak{g})$.

Recall the subalgebras \mathfrak{g}_1 and \mathfrak{g}_{-1} of \mathfrak{g} . Let $U(\mathfrak{g}_1)$ and $U(\mathfrak{g}_{-1})$ be their enveloping algebras, and $u(\mathfrak{g}_1)$ and $u(\mathfrak{g}_{-1})$ their images in $u_{\chi}(\mathfrak{g})$.

Lemma 5.1. *There is a \mathbb{F} -vector space isomorphism:*

$$u_{\chi}(\mathfrak{g}) \cong U(\mathfrak{g}_{-1}) \otimes u_{\chi}(\mathfrak{g}_{\bar{0}}) \otimes U(\mathfrak{g}_1).$$

Proof. Let I_{χ} (resp. I_{χ}^0) be the two-sided ideals of $U(\mathfrak{g})$ (resp. $U(\mathfrak{g}_{\bar{0}})$) generated by the central elements (cf [7]) $x^p - x^{[p]} - \chi(x)^p$, $x \in \mathfrak{g}_{\bar{0}}$. By the PBW theorem (see [1]) we get

$$U(\mathfrak{g}) \cong U(\mathfrak{g}_{-1}) \otimes U(\mathfrak{g}_{\bar{0}}) \otimes U(\mathfrak{g}_1).$$

Note that the central elements $x^p - x^{[p]} - \chi(x)^p$, $x \in \mathfrak{g}_{\bar{0}}$ are all contained in $U(\mathfrak{g}_{\bar{0}})$. Then we have

$$\begin{aligned} I_{\chi} &= \sum_x U(\mathfrak{g})(x^p - x^{[p]} - \chi(x)^p) \\ &= \sum_x U(\mathfrak{g}_{-1})U(\mathfrak{g}_{\bar{0}})U(\mathfrak{g}_1)(x^p - x^{[p]} - \chi(x)^p) \\ &= U(\mathfrak{g}_{-1}) \sum_x U(\mathfrak{g}_{\bar{0}})(x^p - x^{[p]} - \chi(x)^p)U(\mathfrak{g}_1) \\ &= U(\mathfrak{g}_{-1})I_{\chi}^0 U(\mathfrak{g}_1) \\ &\cong U(\mathfrak{g}_{-1}) \otimes I_{\chi}^0 \otimes U(\mathfrak{g}_1), \end{aligned}$$

which gives

$$\begin{aligned} u_{\chi}(\mathfrak{g}) &\cong U(\mathfrak{g}_{-1}) \otimes U(\mathfrak{g}_{\bar{0}}) \otimes U(\mathfrak{g}_1) / U(\mathfrak{g}_{-1}) \otimes I_{\chi}^0 \otimes U(\mathfrak{g}_1) \\ &\cong U(\mathfrak{g}_{-1}) \otimes u_{\chi}(\mathfrak{g}_{\bar{0}}) \otimes U(\mathfrak{g}_1). \end{aligned}$$

□

It follows from the lemma that $u(\mathfrak{g}_{\pm 1}) \cong U(\mathfrak{g}_{\pm 1})$.

To study the representations of $u_\chi(\mathfrak{g})$, by applying the automorphisms of the Lie superalgebra \mathfrak{g} we may assume $\chi(e_{ij}) = 0$ for all $(i, j) \in \mathcal{I}_0$ (see [8]). Let M be a simple $u_\chi(\mathfrak{g}_{\bar{0}})$ -module. Then M contains a maximal vector v^+ of weight μ for some $\mu \in H^*$ (cf. [2]). Since M is simple, M is generated by v^+ . Denote M by $M(\mu)$. We view $M(\mu)$ as a $u_\chi(\mathfrak{g}^+)$ -module annihilated by \mathfrak{g}_1 .

Set

$$K_\chi(\mu) = u_\chi(\mathfrak{g}) \otimes_{u_\chi(\mathfrak{g}^+)} M(\mu).$$

Let us note that the maximal vector and the weight μ need not be unique.

Let $\pi : U(\mathfrak{g}) \rightarrow u_\chi(\mathfrak{g})$ be the canonical epimorphism. Then π maps $U(\mathfrak{g}^+)$ (resp. $U(\mathfrak{g}^-)$; $U(\mathfrak{g}_{\bar{0}})$) onto $u_\chi(\mathfrak{g}^+)$ (resp. $u_\chi(\mathfrak{g}^-)$; $u_\chi(\mathfrak{g}_{\bar{0}})$). Using the epimorphism $M(\lambda)$ can be viewed as a $U(\mathfrak{g}^+)$ -module annihilated by \mathfrak{g}_1 . Define the \mathbb{F} -linear mapping

$$\phi : K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^+)} M(\lambda) \rightarrow K_\chi(\lambda)$$

such that $\phi(u \otimes x) = \pi(u) \otimes x$ for all $u \in U(\mathfrak{g})$, $x \in M(\lambda)$. It is easily seen that ϕ is a $U(\mathfrak{g})$ -module epimorphism. By comparing dimensions we obtain that ϕ is an isomorphism.

For $e_{\mathcal{I}_1}, f_{\mathcal{I}_1} \in U(\mathfrak{g})$, let us denote their images in $u_\chi(\mathfrak{g})$ also by the same notation. Let v_λ be a maximal vector in the $u_\chi(\mathfrak{g}^+)$ -module $M(\lambda)$ of weight λ . Then by applying the isomorphism ϕ we see that in $K_\chi(\lambda)$

$$\begin{aligned} e_{\mathcal{I}_1} f_{\mathcal{I}_1} \otimes v_\lambda &= \phi(e_{\mathcal{I}_1} f_{\mathcal{I}_1} \otimes v_\lambda) \\ &= \phi(\Pi_{\alpha \in \Phi_1^+}(\lambda + \rho, \alpha) v_\lambda) \\ &= \Pi_{\alpha \in \Phi_1^+}(\lambda + \rho)(h_\alpha) v_\lambda. \end{aligned}$$

For any $u_\chi(\mathfrak{g}_{\bar{0}})$ -module M , M is viewed as a $u_\chi(\mathfrak{g}^+)$ -module by letting $\mathfrak{g}_1 M = 0$, we define the induced functor from the categories of $u_\chi(\mathfrak{g}_{\bar{0}})$ -modules to the categories of $u_\chi(\mathfrak{g})$ -modules by

$$\text{Ind}(M) = u_\chi(\mathfrak{g}) \otimes_{u_\chi(\mathfrak{g}^+)} M.$$

Clearly Ind is an exact functor and $\text{Ind}(M) = K_\chi(\lambda)$ in case M is a simple $u_\chi(\mathfrak{g}_{\bar{0}})$ -module $M(\lambda)$.

For any $u(\mathfrak{g}_1)$ -module $N = N_{\bar{0}} \oplus N_{\bar{1}}$, let us denote

$$N^{\mathfrak{g}_1} = \{x \in N \mid gx = 0 \text{ for any } g \in \mathfrak{g}_1\}.$$

Lemma 5.2. *For the left regular $u(\mathfrak{g}_1)$ -module $u(\mathfrak{g}_1)$, we have $u(\mathfrak{g}_1)^{\mathfrak{g}_1} = \mathbb{F}e_{\mathcal{I}_1}$.*

Proof. Clearly we have $e_{\mathcal{I}_1} \in u(\mathfrak{g}_1)^{\mathfrak{g}_1}$. Since $u(\mathfrak{g}_1) \cong U(\mathfrak{g}_1)$, $u(\mathfrak{g}_1)$ has a basis e_I , $I \subseteq \mathcal{I}_1$. For any nonzero $x = \sum c_I e_I \in u(\mathfrak{g}_1)$, if there is $I \subsetneq \mathcal{I}_1$ such that $c_I \neq 0$, then we have $e_{ij}x \neq 0$ for some $(i, j) \in \mathcal{I}_1$. This gives $u(\mathfrak{g}_1)^{\mathfrak{g}_1} = \mathbb{F}e_{\mathcal{I}_1}$ \square

Theorem 5.3. *If $\chi(h_\alpha) \neq 0$ for all $\alpha \in \Phi_1^+$, then $u_\chi(\mathfrak{g}_{\bar{0}})$ and $u_\chi(\mathfrak{g})$ are Morita equivalent.*

First we show that $K_\chi(\lambda)$ is simple for any $\lambda \in H^*$. Let $N = N_{\bar{0}} \oplus N_{\bar{1}}$ be a nonzero submodule of $K_\chi(\lambda)$. Take a nonzero elements $v \in N$, by applying appropriate $f_{ij}((i, j) \in \mathcal{I}_1)$ we get $f_{\mathcal{I}_1} \otimes x \in N$ for some $0 \neq x \in M(\lambda)$. We may assume x is a weight vector, i.e., $x \in M(\lambda)_\mu$ for some $\mu \in H^*$. Since $M(\lambda)$ is a simple $u_\chi(\mathfrak{g}_{\bar{0}})$ -module, we have $u_\chi(\mathfrak{g}_{\bar{0}})x = M(\lambda)$. Hence, there is an element

$$f = \sum c_i u_i^- u_i^0 u_i^+ \in u_\chi(\mathfrak{g}_{\bar{0}})$$

such that $fx = v_\lambda$, where u_i^- (resp. u_i^+ ; u_i^0) is the product of f_{ij} (resp. e_{ij} ; e_{ss}), $(i, j) \in \mathcal{I}_0$, $1 \leq s \leq m+n$, $c_i \in \mathbb{F}$ and v_λ is a maximal vector of the weight λ .

Since x is a weight vector, we may assume $f = \sum c_i u_i^- u_i^+$. Since each e_{ij} and f_{ij} with $(i, j) \in \mathcal{I}_0$ commutes with $f_{\mathcal{I}_1}$, by applying f to $f_{\mathcal{I}_1} \otimes x \in N$ we get $f_{\mathcal{I}_1} \otimes v_\lambda \in N$. Applying $e_{\mathcal{I}_1}$ to which we get

$$\Pi_{(i,j) \in \mathcal{I}_1}(\lambda + \rho, \epsilon_i - \epsilon_j) v_\lambda = \Pi_{\alpha \in \Phi_1^+}(\lambda + \rho)(h_\alpha) v_\lambda \in N.$$

Note that $h_\alpha^p - h_\alpha^{[p]} - \chi(h_\alpha)^p$ in $u_\chi(\mathfrak{g})$, so that

$$\lambda(h_\alpha)^p - \lambda(h_\alpha) = \chi(h_\alpha)^p,$$

which implies that $\lambda(h_\alpha) \notin \mathbb{F}_p$. Since $\rho(h_\alpha) = (\rho, \alpha) \in \mathbb{F}_p$ for any $\alpha \in \Phi_1^+$, we get $\Pi_{\alpha \in \Phi_1^+}(\lambda + \rho)(h_\alpha) \neq 0$, which gives $v_\lambda \in N$. Therefore $N = K_\chi(\lambda)$, so that $K_\chi(\lambda)$ is simple.

Next we show that $K_\chi(\lambda)^{\mathfrak{g}_1} = M(\lambda)$. Note that the subspace $f_{\mathcal{I}_1} \otimes M(\lambda) \subseteq K(\lambda)$ is annihilated by \mathfrak{g}_{-1} . Since $e_{ij}, f_{ij}, (i, j) \in \mathcal{I}_0$ commutes with $f_{\mathcal{I}_1}$, the subspace is a simple $u_\chi(\mathfrak{g}^-)$ -submodule of $K_\chi(\lambda)$. Since $K_\chi(\lambda)$ is simple, we have

$$\begin{aligned} K_\chi(\lambda) &= u_\chi(\mathfrak{g}) f_{\mathcal{I}_1} \otimes M(\lambda) \\ &= u(\mathfrak{g}_1) u_\chi(\mathfrak{g}_{\bar{0}}) u(\mathfrak{g}_{-1}) f_{\mathcal{I}_1} \otimes M(\lambda) \\ &= u(\mathfrak{g}_1) f_{\mathcal{I}_1} \otimes M(\lambda). \end{aligned}$$

Set

$$K_\chi^-(f_{\mathcal{I}_1} \otimes M(\lambda)) = u_\chi(\mathfrak{g}) \otimes_{u_\chi(\mathfrak{g}^-)} (f_{\mathcal{I}_1} \otimes M(\lambda)),$$

where $f_{\mathcal{I}_1} \otimes M(\lambda)$ is viewed as a $u_\chi(\mathfrak{g}^-)$ -module annihilated by \mathfrak{g}_{-1} . By the comparison of dimensions we have that $K_\chi(\lambda)$ is isomorphic to $K_\chi^-(f_{\mathcal{I}_1} \otimes M(\lambda))$ as $u_\chi(\mathfrak{g})$ -modules. Thus, as $u(\mathfrak{g}_1)$ -modules, we have

$$K_\chi(\lambda) \cong u(\mathfrak{g}_1) \otimes_{\mathbb{F}} f_{\mathcal{I}_1} \otimes M(\lambda),$$

from which it follows that

$$\begin{aligned} K_\chi(\lambda)^{\mathfrak{g}_1} &\cong u(\mathfrak{g}_1)^{\mathfrak{g}_1} \otimes f_{\mathcal{I}_1} \otimes M(\lambda) \\ &\cong e_{\mathcal{I}_1} f_{\mathcal{I}_1} \otimes M(\lambda) \\ &= M(\lambda), \end{aligned}$$

where the last equality is given by the fact that $e_{\mathcal{I}_1} f_{\mathcal{I}_1} v_\lambda \neq 0$.

From above discussion, we have that the functor $(,)^{\mathfrak{g}_1}$ is right adjoint to Ind . By a similar argument as that for [2, Th. 3.2], $u_\chi(\mathfrak{g}_{\bar{0}})$ and $u_\chi(\mathfrak{g})$ are Morita equivalent.

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